Abstract of doctoral dissertation:

Analysis of Pairwise Comparison Matrix Inconsistency Characteristics and Their Relationships with Priority Vector Estimates Quality

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My doctoral dissertation concerns one of the most widely used method of multicriteria analysis - the *Analytic Hierarchy Process* (AHP). The dissertation starts with a brief presentation the origin and the theoretical basis of this method. Next, I conduct the simulation-based research exploring the main foundation of its methodology. In particular, I focus on the inconsistency analysis, which is an important part of the AHP. I present definitions of the most popular inconsistency indices and examine their relationship with the errors in priority vector estimation.

In Chapter I, the general foundation of the contemporary decision theory, specifically of the multicriteria decision-making theory is recalled. In general, the decision theory can be broken into two main branches: normative and descriptive theory. The second one analyzes and describes people's behaviors in real-world decisive situation. It appears, that those behaviors often seems to be inconsistent and unreasonable. To explain such situation and provide methods to deal with them, the descriptive part of decision theory uses psychological and sociological studies and methods. In contrast, normative theory primarily deals with mathematical models of real decision making problems and provides us with both the concepts of possible solution and methods to resolve them. In this thesis I deal with the normative theory, thus in the first chapter, I describe the main ideas and concepts of this branch of decision theory. A special part of normative decision theory is multicriteria decision analysis. Multicriteria decisions are naturally more complicated and difficult for decision makers in praxis, since each alternative or decision needs to be evaluated with regard to various, often conflicting criteria. Thus such problems attracted the attention researches and practitioners for many years, and consequently, various techniques which support decision-makers in such problems have been developed over the last decades. Among them is the AHP - decision making methodology that is based on pairwise comparisons

of both alternatives and criteria. Pairwise-comparison-based inference had been naturally used in previous centuries, but in AHP, its founder T. L. Saaty has connected this technique with hierarchical structure of the alternatives and criteria and named it Analytic Hierarchy Process.

Chapter II presents a brief discussion of the epistemological and cognitive importance of computer simulation. The growing power of computers and constantly developed simulation methodology have led to recognition of simulation as an important method for advancing the natural, social, and engineering sciences. Many simulation models have a so-called black-box structure. In a black-box concept, the simulation model transforms the observable inputs into observable outputs, while the whole transformation process codded in simulation's computer modules as well as various internal variables are unobservable. The result presented in this thesis are obtained in this type of simulation model. The input date (e.g. true priority vector), as well as all random perturbation of the pairwise comparison matrix, actually was generated by computer. This kind of simulation in turn is called Monte Carlo simulation. In Monte Carlo simulation the complex phenomenon is explored in simplified way. We can simulate the behavior of phenomenon arbitrarily many times and then analyze the results with the help of statistical inference methods. Obvious, we have to make sure the size and variety of input data are big enough and the values of random data are generated from the proper probability distribution so that they are representative and properly mimic the underlying phenomenon.

Chapter III concerns the AHP methodology and its possible modifications. I define formally the most important notions such as *priority vector*, *pairwise comparison matrix*, *consistent matrix* or *priority ratios judgment scale*. In this chapter, some results of simulations are presented as well, thanks to which we can assess the significance of the adopted judgment scale or the adopted prioritization procedure. Since in the literature some alternative methods of calculating the priority vectors based on pairwise comparison matrix are presented, I also compare these methods in my simulation experiments. In my research, various shapes ans characteristics of matrix errors' distribution were taken into account, and their impact on the final results was studied. In particular, I present average values, standard deviations and ranges (max and min) of absolute (AE) and relative (RE) estimation errors for such different setups. Additionally, in Chapter III, I also examine *ordering errors* (OE) and *significantly wrong final ranking* (SWFR) errors. These kind of errors are of special importance in the problems of *the choice of the best alternative*. One of the important conclusion from this part of my research was the choice of the prioritization method for my further experimental studies. Taking into account the simulation results and number of arguments from literature, I decided to use the geometric mean procedure in this dissertation.

The fourth and fifth chapters of my work is devoted to inconsistency analysis. The analysis of inconsistency is an important part of AHP. Pairwise comparison matrices that are used in this procedure, are the one and only source of information about the decision maker's preferences. That is why it is so important to make sure there are no significant errors in these matrices. However it is well understood that such errors, or maybe more adequately the inconsistencies of the judgment included in pairwise comparison matrices usually occur. These inconsistencies are not always solely the result of decision maker's mistakes or uncertainty of the judgments, but sometimes they can be entailed by procedure itself. For example, a decision maker's opinions about his/her preferences are converted into a number from limited set, thus the origin of such errors could be the scale limits. In my research, I take into account various aspects related to the inconsistency of the pairwise comparison matrix and examine their relationship with priority estimation errors. Presented research results confirm that typically if the magnitude of the inconsistencies increase than the magnitude of estimation errors increase as well. However this relation has decidedly statistical character, and the strength of this relationship depends clearly on the adopted inconsistency *measure*. In the AHP, such measures of inconsistency are called *inconsistency indices*. In literature, we can encounter definitions of many such indices. Those of them which I study in my research are presented in the beginning of the Chapter IV. There are two most popular indices in literature: the *Saaty's index* and a *geometric index*. Both of the indices are closely related to specific prioritization procedure. Apart from these two, there are also other interesting indices defined, which are not connected with any specific prioritization procedure. Among them, there is a group of indices that are based on introduced by Koczkodaj's idea of triads and their inconsistency. The most known indices from this group are the *Koczkodaj's index* and *Salo-Hämäläinen's index*. The indices which are based on the idea of triad has some advantages. For example, their values are explicitly related to inconsistencies of some specific elements in pairwise comparison matrices, so this makes it possible to correct the most significant errors in matrix. Apart from the well-known indices in Chapter IV, I also present definitions of some new ones that are also based on the notion of the triad inconsistency. The presented inconsistency indices are then examined in Monte Carlo simulation.

It is quite obvious that a *proper* inconsistency index should be characterized by a *high* correlation with the magnitude of the errors in estimated priority vector. Thus in Chapter IV, I also present results of simulation experiments in which the relationship between inconsistency indices values and various types of estimation errors are examined. For this purpose I simulate framework, in which whole AHP setups are simulated; in these simulation, the values of the *true* priority vectors and perturbations in *true* pairwise comparison matrices are generated. Next, the values of priority-vector-estimates are calculated based on these *disturbed* matrices. Finally, the values of correlation coefficient between inconsistency indices and errors in priority estimates are calculated. I take into account two fundamental characteristics of correlation: the Spearman and Pearson correlation coefficient. Obviously, the Spearman coefficient is more important for us, because the relationship between the index and prioritization errors should be monotonic, but not necessarily linear. However linear relation would be convenient in praxis, because in such a case the growth of one variable would be followed by proportional growth of other variable. The results of my simulations are interesting, it turns out that we observe almost ideal linear relationship with estimation errors for some of the indices under study. These coefficients are presented in tables in Chapter IV. Conducted simulations enable us to study this relationship in two cases - for singular matrices separately and for whole typical AHP setup in which we deal with many such matrices. It appears that even higher correlations are observed in the latter case. Although the values of correlation coefficient apparently depend on the specific indices, all values are high enough for most of them. Slightly lower values are obtained for SWFR errors, but nonetheless, these coefficient also indicate eligible properties of most examined indices. I also examine performance of the indices under various priority scales. It appears that from the investigated relationship standpoint, the adopted scale has rather marginal significance - the simulations that were conducted for scales other than the one devised by Saaty provide the results similar in spirit, or even the correlation was better when the standard Saaty's scale was used.

Chapter V is devoted to the pairwise comparison matrix acceptance procedure. Inconsistency indices have been introduced to AHP especially with the purpose of verifying whether a given pairwise comparison matrix is trustworthy or maybe it contain too many errors and consequently, the resultng priority estimates are misleading and/or useless. Since the correlation coefficient between examined inconsistency indices and values of the errors in priority vector estimate are high, we can use this indices into the matrix acceptance procedure. However, a question arises here: when we should accept the matrix and when it should be reject on the basis of inconsistency index value? The founder of the AHP suggests calculating the value of the average inconsistency index for a number of random reciprocal matrices and to accept matrices that possess an index that is less then ten percent of the calculated random value. Although such a procedure is usually used in practice, it has no formal/statistical justification at all. Therefore in my dissertation, I look for an alternative procedure that is based on sound statistical methodology. The development of such procedure is the goal of Chapter V. In that chapter for four inconsistency indices separately, I build the regression models describing the relationships between their values and the errors in priority vector estimate. For this purpose, I choose indices with the highest Pearson correlation coefficient values. Namely, the indices are as follow: Saaty's index, Koczkodaj's index, ATI (devised by Grzybowski) and MLTI (devised by Kazibudzki).

The final models are presented in tables in Chapter V. As expected on the basis of the Pearson correlation, the models turn out to be very good. Apart from the model coefficient, the tables show additionally two fundamental model-quality characteristics: *model standard error* (MSE) and *coefficient of determination* R^2. It is interesting to observe that the lowest values of R^2 coefficient have been obtained for Saaty's index, but still those values are bigger than 0.8, so even those models seems to be satisfactory. In turn, values of determination coefficient are usually over 0.9 for the models related to remaining indices. Slightly worse results have been obtained for correlation with SWFR values, especially for Koczkodaj index. In turn, the ATI manifests the highest agreement between the model of regression and the results obtained in simulation, with a value of R^2 close to 1 for the regression model for AE. On the graphs presented in this chapter the degree of the agreement between the developed models and the simulation data points can be observed. Because of the limited space of the dissertation the presented graphs are plotted for selected numbers of alternatives n and criteria k. However in tables all values of regression models coefficients for all investigated cases of $n = 3, \ldots, 9$ and $k = 3, \ldots, 7$ are gathered. Because presented results depend clearly on the number of criteria k and alternatives n, I also develop models with two and three explanatory variables, where additional variables are the numbers of criteria and/or alternatives. Models with a larger number of variables usually outperform the simpler ones. However, we should keep in mind that this results are obtained for clustered data, where average n or k would be fraction, so in practice, the agreement would be somewhat less. However in my opinion there is no doubt all presented models can be used in pairwise comparison matrix acceptance procedure.

In the appendix I present regression models for another four indices: *geometric index*, *Salo-Hämäläinen's index* and two new ones that are based on triad notion. The results show that there exists a whole group of indices, which can be used in practice for the assessment of the usefulness of the pairwise comparison matrices.